Student Workbook

Exploring Hyperbolas

Name: ______________________ Date: ______________ Class: ________

ESSENTIAL QUESTION

How does the hyperbola differ from the ellipse in its definition and equation, and what role do the asymptotes play in graphing?

Lesson Overview

A hyperbola is the set of all points where the DIFFERENCE of distances to two fixed points (foci) is constant, equal to 2a: $|PF_1 - PF_2| = 2a$. Unlike the ellipse, a hyperbola has two separate branches and a pair of asymptotes that guide its shape.

$$x^2/a^2 - y^2/b^2 = 1 \text{ (horizontal)}$$
$$y^2/a^2 - x^2/b^2 = 1 \text{ (vertical)}$$
$$c^2 = a^2 + b^2$$

Standard Form	Horizontal (opens left/right): $x^2/a^2 - y^2/b^2=1$, foci $(\pm c,0)$, asymptotes $y=\pm(b/a)x$. Vertical (opens up/down): $y^2/a^2 - x^2/b^2=1$, foci $(0,\pm c)$, asymptotes $y=\pm(a/b)x$.
Critical Difference	For hyperbolas $c^2 = a^2 + b^2$ (NOT $a^2 - b^2$ like ellipses). The foci are OUTSIDE the vertices, so $c > a$.
Vertices	On the transverse axis, distance a from the center — always ON the hyperbola.
Asymptotes	Lines the hyperbola approaches but never touches. Box method: draw a $2a \times 2b$ rectangle centered at (h,k); the diagonals are the asymptotes.
Eccentricity	$e = c/a > 1$ always (unlike the ellipse, where $0<e<1$). Larger e = flatter, more open branches.

Worked Examples

EXAMPLE 1

Identify all features of $x^2/9 - y^2/16 = 1$.

- $a^2 = 9 \rightarrow a = 3$; $b^2 = 16 \rightarrow b = 4$
- $c^2 = 9 + 16 = 25 \rightarrow c = 5$
- Horizontal hyperbola (positive x^2 term)

Answer: Center (0,0), vertices (±3,0), foci (±5,0), asymptotes $y = \pm(4/3)x$, $e = 5/3$

EXAMPLE 2

Write the equation of a hyperbola with vertices (±2, 0) and foci (±√13, 0).

- Horizontal hyperbola; $a = 2 \rightarrow a^2 = 4$; $c = \sqrt{13} \rightarrow c^2 = 13$
- $b^2 = c^2 - a^2 = 13 - 4 = 9$

Answer: $x^2/4 - y^2/9 = 1$

EXAMPLE 3

Find all features of $(x+1)^2/4 - (y-3)^2/9 = 1$.

- Center $(-1,3)$; $a^2 = 4 \to a = 2$; $b^2 = 9 \to b = 3$
- $c^2 = 4 + 9 = 13 \to c = \sqrt{13}$; horizontal
- Vertices: $(-1\pm2, 3) = (1,3), (-3,3)$. Asymptotes: $y - 3 = \pm(3/2)(x+1)$

Answer: Center $(-1,3)$, vertices $(1,3)$ & $(-3,3)$, foci $(-1\pm\sqrt{13}, 3)$

EXAMPLE 4

Identify all features of $y^2/25 - x^2/144 = 1$.

- Vertical hyperbola (y^2 term is positive)
- $a^2 = 25 \to a = 5$; $b^2 = 144 \to b = 12$
- $c^2 = 25 + 144 = 169 \to c = 13$

Answer: Vertical hyperbola; vertices $(0,\pm5)$, foci $(0,\pm13)$, asymptotes $y = \pm(5/12)x$

EXAMPLE 5

Convert $4x^2 - 9y^2 - 16x + 54y - 101 = 0$ to standard form.

- Group: $(4x^2 - 16x) - (9y^2 - 54y) = 101$
- Factor: $4(x^2 - 4x) - 9(y^2 - 6y) = 101$
- Complete the square: $4(x-2)^2 - 16 - 9(y-3)^2 + 81 = 101 \to 4(x-2)^2 - 9(y-3)^2 = 36$
- Divide by 36

Answer: $(x-2)^2/9 - (y-3)^2/4 = 1$; center $(2,3)$, $a=3$, $b=2$

Guided Practice

Work through each problem below. Use the hint if you get stuck — write your final answer on the last line.

1 Identify all features of $x^2/36 - y^2/13 = 1$.

Hint: $c^2 = a^2 + b^2$ for hyperbolas (not minus!). Find a, b, c, then state vertices, foci, and asymptotes.

2 Write the equation of a hyperbola with vertices $(0, \pm4)$ and foci $(0, \pm5)$.

Hint: Vertical hyperbola (vertices on y-axis). Use $b^2 = c^2 - a^2$.

3 Find the asymptotes of $(x-2)^2/16 - (y+1)^2/9 = 1$.

Hint: Asymptotes pass through the center (h,k) with slopes $\pm b/a$.

4 Convert $9y^2 - 4x^2 - 18y + 24x - 63 = 0$ to standard form.

Hint: Group y and x terms separately. Factor out 9 and −4 before completing the square.

5 A hyperbola has asymptotes $y = \pm(3/2)x$ and passes through (2, 3). What can you conclude?

Hint: Check whether (2, 3) actually satisfies $y = (3/2)x$ first.

Key Vocabulary

Hyperbola	Set of all points where the absolute difference of distances to two foci equals 2a.
Transverse axis	The axis connecting the two vertices; length 2a.
Vertices	Endpoints of the transverse axis, distance a from center.
Foci	Two fixed points; $c^2 = a^2 + b^2$ (foci are outside the vertices).
Asymptotes	Lines the hyperbola approaches but never touches; $y-k = \pm(b/a)(x-h)$ for a horizontal hyperbola.
Eccentricity	$e = c/a > 1$ for all hyperbolas.

Quick Check Quiz

Circle the letter of the correct answer for each question.

1 **For $x^2/25 - y^2/144 = 1$, the foci are at:** Multiple Choice

A (±5, 0)
B (±12, 0)
C (±13, 0)
D (0, ±13)

2 **The asymptotes of $x^2/4 - y^2/9 = 1$ are:** Multiple Choice

A $y = \pm(2/3)x$
B $y = \pm(3/2)x$
C $y = \pm(2/9)x$
D $y = \pm(9/4)x$

3 **Which equation represents a vertical hyperbola?** Multiple Choice

A $x^2/9 - y^2/4 = 1$
B $y^2/9 - x^2/4 = 1$
C $x^2/9 + y^2/4 = 1$
D $x^2/4 - y^2/9 = 1$

4 **For a hyperbola, c^2 equals:** Multiple Choice

- **A** $a^2 - b^2$
- **B** $a^2 + b^2$
- **C** $b^2 - a^2$
- **D** $\sqrt{(a^2 + b^2)}$

5 **The eccentricity of a hyperbola is always:** Multiple Choice

- **A** Between 0 and 1
- **B** Equal to 1
- **C** Greater than 1
- **D** Less than 0

Common Mistakes to Avoid

✗ Using $c^2 = a^2 - b^2$ for hyperbolas (like ellipses).

✓ For hyperbolas: $c^2 = a^2 + b^2$. The foci are OUTSIDE the vertices, so $c > a$.

✗ Thinking the asymptotes touch or cross the hyperbola.

✓ Asymptotes are lines the hyperbola approaches but NEVER touches. They are guides for graphing, not part of the curve.

✗ Confusing which term is a^2 in a vertical hyperbola.

✓ In $y^2/a^2 - x^2/b^2 = 1$, a^2 is under y^2 (the positive term). The vertices are on the y-axis at $(0, \pm a)$.

✗ Writing asymptote slopes as $\pm a/b$ for a horizontal hyperbola.

✓ For $x^2/a^2 - y^2/b^2 = 1$, asymptotes are $y = \pm(b/a)x$ — b is in the numerator, a in the denominator.

Math Tips

★ The key difference: ellipse has + between terms ($x^2/a^2+y^2/b^2=1$); hyperbola has − ($x^2/a^2-y^2/b^2=1$).

★ Asymptote box trick: draw a rectangle with width 2a and height 2b centered at (h,k). The diagonals of this box are the asymptotes.

★ For hyperbolas $c^2 = a^2 + b^2$, so c is always larger than both a and b. The foci are always outside the vertices.

★ To identify orientation: positive x^2 term → opens left/right; positive y^2 term → opens up/down.

★ Eccentricity $e > 1$ for hyperbolas. The larger e is, the more 'open' (flat) the hyperbola looks.